in cases where a wide difference in signal amplitudes exists [305]. They use a Serrodyne technique to translate both signals to audio, where wide differences in level can readily be handled. Another analysis compares phase-shift mismatch errors for several choices of reference wave [306]. Hu has extended the modulated dipole scatterer method for measuring electric field to the case of a modulated loop scatterer for measuring magnetic field [307]. Other devices include a calorimeter with an absorptive harmonic filter for multimode power measurements [308], and a bridge-type impedance meter [309].

[303] J. K. Hunton, "Analysis of microwave measurement techniques by means of signal flow graphs," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 206–212; March, 1960.
[304] D. S. Lerner and H. A. Wheeler, "Measurement of bandwidth of microwave resonator by phase shift of signal modulation," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 343–345; May, 1960.
[305] C. A. Finnila, *et al.*, "Measurement of relative phase shift at microwave frequencies," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 143–147; March, 1960.
[306] G. E. Schafer, "Mismatch errors in microwave phase shift measurements," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 617–622; November, 1960.
[307] M.-K. Hu, "On measurements of microwave $\bar{E}$ and $\bar{H}$ field distributions by using modulated scattering methods," IRE Trans. on Microwave Theory and Techniques, vol. MTT-8, pp. 295–300; May, 1960.
[308] V. G. Price, "Harmonic calorimeter for power measurements in a multimode waveguide," 1960 IRE International Convention Record, pt. 3, pp. 136–144.
[309] R. W. Beatty, "A microwave impedance meter capable of high accuracy," IRE Trans. on Microwave Theory and Techniques (Correspondence), vol. MTT-8, pp. 461–463; July, 1960.

Several new techniques have appeared for measuring specific devices or material characteristics. In the Doppler method for measuring back scatter, the sample (in this case absorber) is nutated by means of counter-rotating eccentric disks in a ground plane [310]. This imparts an audio modulation to the scattered return, thereby allowing greater suppression of extraneous signals. A nanosecond-pulse radar has been useful in identifying internal reflections of TWT's [311]. Several papers relate to excess carrier lifetime in semiconductor materials [312]–[314]. The technique (see Section IV-C) uses a section of waveguide filled with the material; incident light pulses create excess carriers whose lifetime is determined from measurements of the microwave power absorption through the sample.

[310] E. B. McMillan, and H. J. Schmitt, "Doppler method for absorber testing," *Microwave J.*, vol. 3, pp. 64–68; November, 1960.
[311] D. O. Melroy and H. T. Closson, "Measurement of internal reflections in traveling-wave tubes using a millimicrosecond pulse radar," Proc. IRE, vol. 48, pp. 165–168; February, 1960.
[312] H. Jacobs, *et al.*, "Further consideration of bulk lifetime measurement with a microwave electrodeless technique," Proc. IRE, vol. 48, pp. 229–233; February, 1960.
[313] ——, *et al.*, "Some device aspects of multiple microwave reflections in semiconductors," 1960 IRE WESCON Convention Record, pt. 3, pp. 42–48.
[314] H. A. Atwater, "Microwave measurement of semiconductor carrier lifetimes," *J. Appl. Phys.*, vol. 31, pp. 938–939; May, 1960.

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The Short Pulse Behavior of Lossy Tapered Transmission Lines*

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Summary—**An analytic method is given which allows the design engineer to assess rapidly the short pulse characteristics of any given tapered-transmission-line type of pulse transformer. The method allows inclusion of both skin-effect losses and losses which are independent of frequency. The effects of mismatching at either end are shown to be as important as the taper function of the line itself. The results of this approximate method are expressed as simple integrals and matching terms to which it is easy to attach physical significance.**

The method is applied to the analysis of two tapered-line pulse transformers which are geometrically uniform coaxial structures with tapered dielectric constants. The line whose nominal characteristic impedance is an exponential function of electrical position is shown to have a good rise time and tilt distortion characteristics.

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Introduction

In the already extensive literature on the tapered-transmission-line pulse transformer,[1] there has been little investigation of the pulse-distorting effects of losses in the tapered line. That such distortions must exist is evident, for even the lossy uniform line can be

[1] H. Kaufman, "Bibliography of nonuniform transmission lines," IRE Trans. on Antennas and Propagation (Communication), vol. AP-3, pp. 218–220; October, 1955.

demonstrated to produce distortion if terminated in a pure resistance. The so-called "initial slope" distortion of the tapered transmission line which results from the high-pass nature of the ideal device would be expected to be changed to a band-pass characteristic by skin effect. A band-pass filter would, in turn, be expected to lengthen the rise time of a short pulse passing through it. This paper will concern itself with an investigation of the role of losses—primarily skin-effect losses—in lengthening the rise time and otherwise distorting the waveform of a short pulse undergoing impedance transformation by a tapered-transmission-line pulse transformer. Analytical expressions are found which relate pulse waveform distortion to the parameters of the line and to the degree of matching achieved at each end of the line. These expressions do not include the effects of multiple reflections from the ends of the line, since the effects of such additional reflections do not manifest themselves until long after the passage of pulses of practically usable length.

The problem of skin effect in the uniform transmission line has already been considered by Pélissier.[2] He has shown that skin effect can be introduced by adding a series impedance term which is proportional to the square root of frequency. The real part of this impedance is the familiar skin resistance.

The analysis is made by first introducing skin effect into the telegraph differential equations and Laplace transforming them with respect to time. A series solution for the voltage e on the lossy tapered transmission line is developed. A similar series technique is then used to find the impedance Z of the line. The boundary conditions at the generator and load ends of the tapered line are introduced through the use of transfer functions similar to the transmission coefficients of conventional uniform line analysis. An over-all transfer function which contains integrals of the tapered line's parameters over distance x is then developed as a function of s. The inverse Laplace transformation is examined for the case of a step function generator input and used to write down relatively simple expressions for the rise time, tilt, time delay and voltage multiplication ratio of the arbitrarily tapered lossy transmission-line pulse transformer driven through and terminated in arbitrary generator and load resistances. These rise time and tilt distortions are illustrated in Figs. 1 and 2 for the case of a voltage step function applied to the input of the tapered line.

The Analysis

The starting point of the analysis is the pair of generalized telegraph equations Laplace transformed with respect to time.

$$\dot{e} = -i[Z_{\text{series}}], \tag{1}$$

[2] R. Pélissier, "La propagation des ondes transitoires et périodiques le long des lignes électriques," *Rev. Gén. Élec.*, vol. 59, pp. 379–399, September, 1950; pp. 437–454, October, 1950; pp. 502–512, November, 1950.

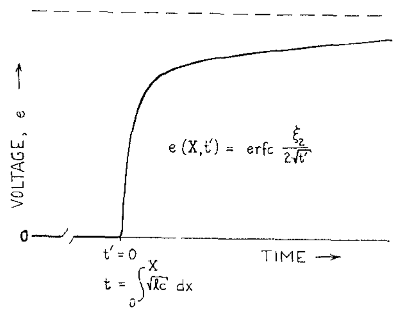

Fig. 1—Skin effect induced rounding of leading edge of pulse on a tapered transmission line.

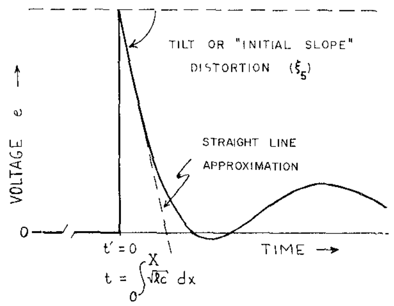

Fig. 2—The same pulse as in Fig. 1, on a greatly contracted time scale.

where

$$\dot{e} = \frac{\partial e}{\partial x},$$

$$\dot{i} = -e[Y_{\text{shunt}}], \tag{2}$$

where

$$\dot{i} = \frac{\partial i}{\partial x}.$$

The series impedance per unit length Z_{series} can be expressed after Laplace transformation with respect to time as $ls+\rho\sqrt{S}+r$. The coefficients l, ρ and r are variables with respect to position x only and correspond respectively to the series inductance per unit length, skin effect coefficient per unit length, and the dc resistance per unit length. The complex skin effect impedance per unit length $\rho\sqrt{S}$ is an exact representation only for the case of plane conductors, but it can be demonstrated that the error introduced in applying this approximation to conductors with other geometries has a negligible effect on the accuracy of the total Z_{series} term when the final results of the analysis are considered.

The shunt admittance per unit length Y_{shunt} is given as the conventional $cs+g$ with both c and g being variables with respect to position only.

Solving the two telegraph equations simultaneously, one obtains a differential equation in voltage.

$$\begin{pmatrix}(l^2c)s^3+(2\rho lc)s^{5/2}+(2rlc+\rho^2c+gl^2)s^2+(2r\rho c+2\rho gl)s^{3/2}\\ +(r^2c+2rgl+\rho^2g)s+(2rg\rho)s^{1/2}+(r^2g)\end{pmatrix}e$$
$$+(\dot{l}s+\dot{\rho}\sqrt{s}+\dot{r})\dot{e}-(ls+\rho\sqrt{s}+r)\ddot{e}=0. \tag{3}$$

A solution to this equation is assumed of the form:

$$\frac{e(x)}{e(0)} = e^{-\Gamma_e(x,s)}. \tag{4}$$

The substitution of (4) into the differential equation in voltage yields a Riccati equation in $\dot{\Gamma}_e$.

$$(ls+\rho\sqrt{s}+r)\dot{\Gamma}_e^2+(\dot{l}s+\dot{\rho}\sqrt{s}+\dot{r})\dot{\Gamma}_e-(ls+\rho\sqrt{s}+r)\ddot{\Gamma}_e$$
$$=(l^2c)s^3+(2\rho lc)s^{5/2}+(2rlc+\rho^2c+gl^2)s^2$$
$$+(2r\rho c+2\rho gl)s^{3/2}+(r^2c+2rgl+\rho^2g)s+(2rg\rho)s^{1/2}+r^2g. \tag{5}$$

Since this type of equation cannot be solved in closed form, a series solution is found by assumption of a series expansion of $\dot{\Gamma}_e$ in decreasing powers of $s^{1/2}$. This form of asymptotic series expansion has been described by Weber and others.[3]

$$\dot{\Gamma}_e = b_1s + b_2s^{1/2} + b_3 + b_4s^{-1/2} + b_5s^{-1} + \cdots. \tag{6}$$

The b coefficients of this series are evaluated after the insertion of the series representation of $\dot{\Gamma}_e$ into (5) by a recursion process. The coefficients of the terms in the highest power of s present are equated to evaluate b_1. Then the terms in the next higher power of s are used together with b_1 to evaluate b_2. This process, when continued, will evaluate the b coefficients in sequence.

There are two independent sets of valid b coefficients corresponding to $m=+1$ and $m=-1$.

$$b_1 = m\sqrt{lc}, \tag{7}$$

$$b_2 = m\frac{\rho}{2}\sqrt{\frac{c}{l}}, \tag{8}$$

$$b_3 = -\frac{1}{4}\left(\frac{\dot{l}}{l}-\frac{\dot{c}}{c}\right)$$
$$+m\left(\frac{1}{2}r\sqrt{\frac{c}{l}}+\frac{1}{2}g\sqrt{\frac{l}{c}}-\frac{1}{8}\frac{\rho^2}{l}\sqrt{\frac{c}{l}}\right), \tag{9}$$

$b_4 =$ all loss terms containing ρ, r and g, (10)

$$b_5 = \frac{m}{\sqrt{lc}}\left\{\frac{7\dot{l}^2}{32l^2}-\frac{5}{32}\frac{\dot{c}^2}{c^2}-\frac{1}{16}\frac{\dot{l}}{l}\frac{\dot{c}}{c}-\frac{1}{8}\left(\frac{\ddot{l}}{l}-\frac{\ddot{c}}{c}\right)\right\}$$

plus loss terms. (11)

An expression for the impedance of the lossy tapered transmission line $Z=e/i$ is found by our differentiating the defining equation for Z and inserting the values of $\dot{e}$ and $\dot{i}$ from (1) and (2).

$$Z = (cs+g)Z^2 - ls - \rho\sqrt{s} - r. \tag{12}$$

[3] E. Weber, "Linear Transient Analysis—Volume II," John Wiley and Sons, Inc., New York, N. Y., Section 7.4; 1956.

This Riccati equation can be solved by the same technique used to solve (5).

$$Z = f_1 + f_2s^{-1/2} + f_3s^{-1} + \cdots. \tag{13}$$

$$f_1 = m\sqrt{\frac{l}{c}} \tag{14}$$

$$f_2 = \frac{m}{2}\frac{\rho}{\sqrt{lc}} \tag{15}$$

$$f_3 = \frac{1}{4c}\left(\frac{\dot{l}}{l}-\frac{\dot{c}}{c}\right) + \text{loss terms}. \tag{16}$$

It is not necessary to assume such a simple power series solution for Z, should a different form of solution be desired. If the following series is chosen:

$$Z = m\sqrt{\frac{l}{c}}\,\epsilon^{\phi_1s^{-1/2}+\phi_2s^{-1}+\phi_3s^{-3/2}+\cdots}, \tag{17}$$

another valid series representation of the high-frequency behavior of Z is obtained. In this case, the exponential must be expanded before evaluation of the ϕ coefficients, which are:

$$\phi_1 = \frac{\rho}{2l}, \tag{18}$$

$$\phi_2 = \frac{m}{4\sqrt{lc}}\left(\frac{\dot{l}}{l}-\frac{\dot{c}}{c}\right) + \text{loss terms}. \tag{19}$$

The index $m=\pm 1$ in the series solutions for e and Z results from the existence of two mathematically independent solutions which are interpreted physically as the voltages and impedances associated with a wave moving in the plus-x direction for $m=+1$ and with a wave moving in the reverse direction for $m=-1$.

For a complete solution, it is necessary to introduce the boundary conditions for the tapered transmission line at both ends of the line—at $x=0$ where the line is driven by a generator with a source resistance R_g and at $x=X$ where the line is terminated in a load resistance R_L. The method chosen for this is that of solving for a transmission coefficient or transfer function at each end of the line. Each transfer function is then considered as a "matching term" independent of the transmission characteristics of the tapered line proper.

The transfer function at the input is defined as:

$$\frac{\vec{e}(0)}{e_g} = \eta\epsilon^{g_0s^{-1/2}+g_1s^{-1}+\cdots}, \tag{20}$$

where

$$\eta \triangleq \frac{1}{1+\dfrac{R_g}{\sqrt{l/c}\,\big|_{x=0}}}, \tag{21}$$

the infinite frequency transfer function at the input.

The g coefficients are evaluated by simple circuit theory using the Z previously found and evaluating it at $x=0$ with $m=+1$ to obtain the input impedance of the line.

Similarly, the transfer function at the output is defined as

$$\frac{e_L}{\vec{e}(X)} = \lambda \epsilon^{h_0 s^{-1/2} + h_1 s^{-1} + \cdots} \tag{22}$$

where

$$\lambda \triangleq \frac{2}{1 + \dfrac{\sqrt{l/c}\,|_{x=X}}{R_L}}, \tag{23}$$

the infinite frequency transfer function at the output.

The choice of transfer functions containing power series in the arguments of the exponentials is a useful one since it allows the over-all transfer function of the tapered-transmission-line pulse transformer to be expressed comparatively simply.

$$\frac{e_L}{e_g} = \frac{\vec{e}(0)}{e_g} \cdot \frac{\vec{e}(X)}{\vec{e}(0)} \cdot \frac{e_L}{\vec{e}(X)}. \tag{24}$$

The three transfer functions making up the over-all transfer function are multiplied together by simply adding up terms in the arguments of the exponentials. By use of the relationship that

$$\exp\left\{-\frac{1}{4}\int_0^X \left(\frac{\dot{l}}{l} - \frac{\dot{c}}{c}\right) dx\right\} = \sqrt{\frac{\sqrt{l/c}\,|_{x=X}}{\sqrt{l/c}\,|_{x=0}}}, \tag{25}$$

the over-all transfer function for the tapered-line pulse transformer becomes

$$\frac{e_L}{e_g} = \eta\lambda \sqrt{\frac{\sqrt{l/c}\,|_{x=X}}{\sqrt{l/c}\,|_{x=0}}}\, \epsilon^{\xi_1 s + \xi_2 s^{1/2} + \xi_3 + \xi_4 s^{-1/2} + \xi_5 s^{-1} + \cdots}, \tag{26}$$

where

$$\xi_1 = -\int_0^X \sqrt{lc}\, dx, \tag{27}$$

$$\xi_2 = -\frac{1}{2}\int_0^X \rho \sqrt{\frac{c}{l}}\, dx, \tag{28}$$

$$\xi_3 = -\int_0^X \left\{\frac{1}{2} r \sqrt{\frac{c}{l}} + \frac{1}{2} g \sqrt{\frac{l}{c}} - \frac{1}{8}\frac{\rho^2}{l}\sqrt{\frac{c}{l}}\right\} dx, \tag{29}$$

$$\begin{aligned}\xi_4 = {} & \frac{1}{2}\frac{\rho}{l}\eta R_g \sqrt{\frac{c}{l_{x=0}}} - \frac{1}{4}\frac{\rho}{l}\left\{\lambda\frac{\sqrt{l/c}}{R_L}\right\}^3_{x=X} \\ & - \frac{1}{2}\int_0^X \frac{\rho}{l}\left(\frac{1}{2}\frac{\dot{l}}{l} - \frac{1}{2}\frac{\dot{\rho}}{\rho} - \frac{1}{2} r\sqrt{\frac{c}{l}} + \frac{1}{2} g\sqrt{\frac{l}{c}} + \frac{1}{8}\frac{\rho^2}{l}\sqrt{\frac{c}{l}}\right) dx,\end{aligned} \tag{30}$$

$$\begin{aligned}\xi_5 = {} & \eta R_g \left\{\frac{1}{4l}\left(\frac{\dot{l}}{l} - \frac{\dot{c}}{c}\right) + \frac{1}{8}\frac{\rho^2}{l^2}\sqrt{\frac{c}{l}}\left(\eta R_g\sqrt{\frac{c}{l}} - 1\right) + \frac{1}{2}\frac{g}{\sqrt{lc}} - \frac{1}{2}\frac{r}{l}\sqrt{\frac{c}{l}}\right\}\Bigg|_{x=0} \\ & + \Bigg[-\frac{\lambda}{8\sqrt{lc}}\left(\frac{\dot{l}}{l} - \frac{\dot{c}}{c}\right) - \frac{1}{8}\frac{\rho^2}{l}\left\{\frac{1}{2}\left(\lambda\frac{\sqrt{l/c}}{R_L}\right) - \frac{1}{2}\left(\lambda\frac{\sqrt{l/c}}{R_L}\right)^2 - \left(\lambda\frac{\sqrt{l/c}}{R_L}\right)^3 + \frac{1}{4}\left(\lambda\frac{\sqrt{l/c}}{R_L}\right)^6\right\} \\ & + \frac{r}{l}\left\{1 - \frac{1}{4}\left(\lambda\frac{\sqrt{l/c}}{R_L}\right)\right\} + \frac{1}{4}\frac{g}{c}\left(\lambda\frac{\sqrt{l/c}}{R_L}\right)\Bigg]_{x=X} \\ & - \int_0^X \Bigg\{\frac{7}{32}\frac{1}{\sqrt{lc}}\frac{\dot{l}^2}{l^2} - \frac{5}{32}\frac{1}{\sqrt{lc}}\frac{\dot{c}^2}{c^2} - \frac{1}{16}\frac{1}{\sqrt{lc}}\frac{\dot{l}}{l}\frac{\dot{c}}{c} - \frac{1}{8}\frac{1}{\sqrt{lc}}\left(\frac{\ddot{l}}{l} - \frac{\ddot{c}}{c}\right) - \frac{5}{128}\frac{\rho^4}{l^3}\sqrt{\frac{c}{l}} - \frac{1}{4}\frac{\rho^2}{l^2}\frac{\dot{l}}{l} \\ & + \frac{1}{4}\frac{\rho^2}{l^2}\frac{\dot{\rho}}{\rho} - \frac{1}{8}\frac{r^2}{l}\sqrt{\frac{c}{l}} - \frac{1}{4}\frac{\dot{r}}{l} + \frac{1}{4}\frac{r}{l}\frac{\dot{l}}{l} - \frac{1}{8}\frac{g^2}{c}\sqrt{\frac{l}{c}} + \frac{1}{4}\frac{\dot{g}}{c} - \frac{1}{4}\frac{g}{c}\frac{\dot{c}}{c} + \frac{3}{16}\frac{\rho^2 r}{l^2}\sqrt{\frac{c}{l}} \\ & + \frac{1}{4}\frac{rg}{\sqrt{lc}} - \frac{1}{16}\frac{\rho^2 g}{l\sqrt{lc}} + \frac{3}{16}\frac{\rho^2 r}{l^2}\sqrt{\frac{c}{l}}\Bigg\} dx.\end{aligned} \tag{31}$$

If a step function EMF is applied at the generator, the load voltage can be found by using e_L/e_g. In this case, a physical interpretation can be assigned to each of the factors of e_L/e_g by consideration of each factor as a filter acting successively on the input step function.

$\eta\lambda \sqrt{\dfrac{\sqrt{l/c}\,|_{x=X}}{\sqrt{l/c}\,|_{x=0}}}$ is a distortionless multiplying factor representing the nominal voltage transformation ratio of the tapered line and its matching as would be predicted from lossless uniform line theory.

$\epsilon^{\xi_1 s}$ represents a distortionless delay of $-\xi_1$ seconds.

$\epsilon^{\xi_2 s^{1/2}}$ represents the rounding of the leading edge of the step function. The shape of this curve is shown in Fig. 1.

ϵ^{ξ_3} represents a distortionless attenuation due to losses in the line.

$\epsilon^{\xi_4 s^{-1/2}}$ represents a completely negligible modification of the delayed, attenuated and rounded-off step function (see below).

$\epsilon^{\xi_5 s^{-1}}$ represents the tilt or "initial slope" distortion discussed by Frank[4] and Young.[5] The exact inverse Laplace transformation of this term has the form:

$$e(t') = J_0[2\sqrt{-\xi_5 t'}].$$

An extremely good approximation to this is the straight line:

$$e(t') = 1 + \xi_5 t'.$$

The previous discussion has implied that the inverse Laplace transformation of the response of cascaded filters can be found by our multiplying together the time responses of the individual filters. In the general case, this is obviously incorrect and real convolution of the factors is needed. Direct multiplication of the time responses is justified only when only one factor of the complex frequency response is appreciably different from unity at a given complex frequency. This is the case here since the spectral aberration corresponding to the skin-effect-induced rounding of the leading edge of the step function occurs at frequencies which are orders of magnitude higher than those frequency response distortions determining the relatively slow tilt of the top of the step function.

The $\epsilon^{\xi_4 s^{-1/2}}$ term has a value which approaches unity with increasing frequency ($t \to 0$). At the frequencies involved in the spectra of the pulses considered here, this term can still be neglected because it is very close to unity and moreover is essentially divergent. The divergent nature of this term can be demonstrated clearly in the case of the uniform line with skin effect, in which case the contribution of this term leads to physically erroneous results. This explains why in the literature only the $\epsilon^{\xi_2 s^{+1/2}}$ term is used in computing skin effect.[2]

General Results

Other simplifications in the mathematics result when the loss terms of (26) are considered. It can be demonstrated that the presence of ρ in all terms past the ξ_2 term results from the divergent nature of the asymptotic series developed for e. Many of the other terms likewise become negligible for practical tapered-line pulse transformers.

On the basis of the preceding analysis, it is possible to write down simplified expressions of each important modification a unit step function EMF will undergo in passing through a tapered-line pulse transformer. These expressions will be valid for the duration of short pulses of usually acceptable distortion.

Time Delay

As would be expected, this delay is determined by the electrical length of the line and the speed of light.

$$\text{Time delay} = \int_0^X \sqrt{lc}\, dx \text{ (seconds)}. \qquad (32)$$

Voltage Transformation

As defined here, this ratio relates the load voltage amplitude to the generator EMF and includes not only the impedance changing effect of the tapered line but also the effects of matching at each end of the line.

$$\text{Voltage transformation ratio} = \eta\lambda \sqrt{\frac{\sqrt{l/c}\,|_{x=X}}{\sqrt{l/c}\,|_{x=0}}} \cdot \exp\left\{-\frac{1}{2}\int_0^X r\sqrt{\frac{c}{l}}\,dx - \frac{1}{2}\int_0^X g\sqrt{\frac{l}{c}}\,dx\right\}. \qquad (33)$$

The input and output matching terms defined in (21) and (23) are η and λ. The attenuation due to r is completely negligible for practical pulse transformers constructed of self-supporting metal conductors. The attenuation due to g will usually contribute attenuation of no more than a few per cent.

[4] I. A. D. Lewis and F. H. Wells, "Millimicrosecond Pulse Techniques," Pergamon Press, Ltd., London, England, pp. 63–93; 1954.

[5] F. J. Young, E. R. Schatz, and J. B. Woodford, "The optimum transmission-line pulse transformer," *Trans. AIEE*, vol. 79, pp. 220–223; July, 1959.

Lengthening of Rise Time

Fig. 1 shows the skin-effect-induced distortion of the voltage response to a unit step function input. The rise time of this response will be conventionally defined here as the time it takes the voltage response to rise from 10 per cent to 90 per cent of its final value.

$$\text{Rise time} = 31.45\xi_2^2 \text{ (seconds)}, \tag{34}$$

where

$$\xi_2 = -\frac{1}{2}\int_0^X \rho\sqrt{\frac{c}{l}}\,dx. \tag{35}$$

Tilt

The tilt is defined as the normalized slope of the early portion of the load voltage response to a step function input at the generator (see Fig. 2). The tilt depends on both the manner in which the transmission line is tapered and the degree of matching attained at each end of the line.

$$\text{Tilt} = \xi_5\left(\frac{\text{volts}}{\text{volts}}\text{ per second}\right)$$

$$= \eta R_g\left\{\frac{1}{4l}\left(\frac{\dot{l}}{l}-\frac{\dot{c}}{c}\right)\right\}\Bigg|_{x=0} - \frac{\lambda}{8\sqrt{lc}}\left(\frac{\dot{l}}{l}-\frac{\dot{c}}{c}\right)\Bigg|_{x=X}$$

$$-\int_0^X\left\{\frac{1}{\sqrt{lc}}\left(\frac{7}{32}\frac{\dot{l}^2}{l^2}-\frac{5}{32}\frac{\dot{c}^2}{c^2}-\frac{1}{16}\frac{\dot{l}}{l}\frac{\dot{c}}{c}\right.\right.$$

$$\left.\left.-\frac{1}{8}\left(\frac{\ddot{l}}{l}-\frac{\ddot{c}}{c}\right)\right)+\frac{1}{4}\frac{\dot{g}}{c}-\frac{1}{4}\frac{g}{c}\frac{\dot{c}}{c}\right\}dx. \tag{36}$$

The four expressions just derived for time delay, voltage transformation ratio, rise time and tilt represent the simplified results of the analysis. They are valid for the duration of short pulses of usually acceptable distortion. For the special case of constant velocity of propagation or the even more special case of the exponential line where the flare constant $\gamma = \dot{l}/l = -\dot{c}/c$, the four expressions will simplify greatly.

Examples of Two Coaxial Lines with Variable Velocity of Propagation

The type of coaxial tapered line chosen for these examples is somewhat unusual in that it is assumed to be constructed of a 20-foot length of rigid $\frac{3}{4}$-inch copper water pipe and a 20-foot length of no. 25 copper wire. The impedance tapering is accomplished by tapering of the dielectric constant of the dielectric material rather than by variation of the conductor geometry. For the 4-to-1 impedance transformation ratio chosen for these examples, it is necessary to change the dielectric constant over a 16-to-1 range. One way of doing this would be to mix heavily aerated plastic and powdered sintered barium strontium titanate in differing proportions as the line is filled from one end.

Two dielectric constant taper functions will be considered so as to illustrate the effect of taper function on the over-all characteristics of the tapered-line pulse transformer. Each line will be designed to match nominally a 50-ohm generator and a 200-ohm load.

The first taper analyzed will be produced by having the dielectric constant vary linearly with physical position from $\epsilon' = 1.26$ to $\epsilon' = 20.1$.

The second taper analyzed will also have ϵ' vary between 1.26 and 20.1, but in such a manner that the nominal characteristic impedance of the line will vary exponentially with electrical position. A wavefront propagating along this line will see a change in $\sqrt{l/c}$ with time identical to that seen by a wavefront traveling along a true exponential line of the same electrical length. Since the velocity of propagation varies with position, the line will not be exponential with respect to physical position and will therefore be called "electrically exponential."

For both lines, the constant geometry implies a constant l and ρ. Only c and g will vary with position. Dielectric loss is considered proportional to dielectric constant and corresponds to a 100-Mc power factor of 0.1 per cent.

The response of each of the tapered lines to a unit step function generator EMF is determined by use of the simplified expressions (32)–(36). The results are given in Tables I–III (pp. 295 and 296).

TABLE I
PARAMETERS OF THE TWO LINES

	c	g	$\sqrt{l/c}$
	farads per meter	mhos per meter	ohms
Linear taper of dielectric constant	$2.99\cdot10^{-10}(1-0.1539x)$	$1.88\cdot10^{-4}(1-0.1539x)$	$\dfrac{50}{\sqrt{1-0.1539x}}$
Electrically exponential line	$\dfrac{10^{-8}}{(2.840x+5.78)^2}$	$\dfrac{6.283\cdot10^{-3}}{(2.840x+5.78)^2}$	$24.58(x+2.032)$

TABLE II
RESULTS OF THE ANALYSIS

	Delay	Voltage Trans-formation	Rise Time	Tilt (sec^{-1})
Linear taper of dielectric constant	60.6 nsec	0.980	0.67 nsec	$-4.7\cdot10^7$
Electrically exponential line	42.2 nsec	0.987	0.29 nsec	$-4.8\cdot10^6$

It is interesting to note the individual contributions of the three terms making up the tilt term. From (36),

$$\text{tilt} = \begin{pmatrix}\text{sending}\\ \text{end}\\ \text{term}\end{pmatrix} + \begin{pmatrix}\text{load}\\ \text{end}\\ \text{term}\end{pmatrix} + \begin{pmatrix}\text{integral}\\ \text{terms}\end{pmatrix}.$$

TABLE III
TILT TABULATION

	Sending End Term	Load End Term	Integral Terms	Total Tilt
Linear taper of dielectric constant	$+1.28\cdot10^6$	$-8.33\cdot10^7$	$+3.52\cdot10^7$	$-4.7\cdot10^7$
Electrically exponential line	$+8.21\cdot10^6$	$-8.21\cdot10^6$	$-4.8\cdot10^6$	$-4.8\cdot10^6$

The negative tilt is about an order of magnitude less for the nominally matched electrically exponential line than for the nominally matched line with a linear taper of dielectric constant. This is a result of the matching conditions since, as Table III indicates, the linearly-tapered line by itself contributes a positive tilt.

CONCLUSIONS

This investigation has been aimed at examining the role of skin effect in the tapered-line pulse transformer. The approximate mathematical method adopted has fortunately turned out to be general enough also to predict pulse response for time durations very much longer than the duration of the rise phenomenon alone. This has permitted a check with the pulse distortion expressions determined by other investigators for the lossless case. The method can be useful to the design engineer in predicting the entire useful response to a short pulse undergoing impedance transformation on a tapered-transmission-line pulse transformer.

As a result of this analysis, it can be concluded that:

1) The rise time of a tapered transmission line is not materially affected by resistive mismatching at either end of the line.
2) The "initial slope" or tilt of the response to a step function can be adjusted over a large range—including both negative and positive values—depending on the taper function of the line and the generator and load resistances.

This latter conclusion seems to bear out previous contentions that any "optimum" taper would have to be a function of the load and generator impedances.[5]